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Dynamic coherent risk measures

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Abstract

Monetary measures of risk like *Value at Risk* or *Worst Conditional Expectation* assess the risk of financial positions. The existing risk measures are of a static, one period nature. In this paper, I define dynamic monetary risk measures and I present an axiomatic approach that extends the class of *coherent risk measures* to the dynamic framework. The axiom of translation invariance has to be recast as *predictable* translation invariance to account for the release of new information. In addition to the coherency axioms, I introduce the axiom of *dynamic consistency*. Consistency requires that judgements based on the risk measure are not contradictory over time. I show that consistent dynamic coherent risk measures can be represented as the worst conditional expectation of discounted future losses where the expectations are being taken over a set of probability measures that satisfies a consistency condition.

MSC: 91B28; 91B30

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0. Introduction

Consider an institution that has settled on a certain way to measure the risk of its financial positions. As the trading day goes by, changes are made to the position and new information is being released. On the next morning, the institution wishes to reconsider the risk of its changed position taking into account the new information in a

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proper way. A proper way is understood here as a rational way: it is important not to contradict oneself over time in one's risk assessments. The institution asks therefore: how should we process new information and how should we treat changes in the position?

Here, I present one possible way to answer this question by extending the path breaking work by Artzner et al. (1999) (ADEH in the sequel) on *coherent risk measures* to the multiperiod framework. To this end, I adapt the coherency axioms from ADEH to the dynamic setting. In addition, I impose dynamic consistency in the following sense. If two positions are assigned the same risk in all possible states of the world tomorrow, and no payments are due tomorrow for both positions, the two positions should have the same risk today. In particular, if in every future contingency, a position is acceptable, then it should be acceptable today. I show that dynamic risk measures satisfy these axioms if and only if they assign to a sequence (D_t) of payments the risk

$$\rho_t(D) = \max_{P \in \mathscr{P}} \mathbb{E}^{\mathrm{P}} \left[- \sum_{s=t}^{T} \frac{D_s}{(1+r)^{s-t}} \,\middle|\, \mathscr{F}_t \right] \tag{1}$$

for a convex, closed set of probability measures $\mathscr{P}$ on the state space that satisfies a certain consistency condition.

The representation answers the question stated in the first paragraph. Changes in the position are to be taken into account by recalculating the (stochastic) present value of future payments. Information is processed via updating in a Bayesian way every single probability measure in the set $\mathscr{P}$. The new risk of the position is given by the maximal expected present value of future losses where the expectation is taken under the updated probability measures.

In addition, I show that consistency of risk measurement corresponds to a certain closure property of the set $\mathscr{P}$, which I call also consistency. Thus, as *every* set of probability measures qualifies as a possible choice for a coherent risk measure in the static framework, only consistent sets of probability measures yield *dynamically consistent* coherent risk measures. A family of probability measures is consistent if it can be constructed by a backward induction procedure. The procedure can be easily illustrated with a two period example. Say that, in the first period, the random variable X_1 is revealed, and in the second period, you learn the value of the random variable X_2. Suppose that for every possible realization x_1 of X_1 in the first period, you have specified a conditional probability P^{x_1} for the second period random variable X_2 given that $X_1 = x_1$. Moreover, assume that R is the (marginal) probability distribution of X_1. Then one can derive the joint distribution of X_1 and X_2 by the formula for total probability (see formula (2)). A set of probability measures $\mathscr{P}$ is consistent if the set is closed under arbitrary pasting of conditional probabilities and marginal distributions from this set. [2]

[2] The consistency property for a set of probability measures appears before in the literature. It has been called rectangularity by Epstein and Schneider (2003), reduced family by Sarin and Wakker (1998) in a decision theoretic framework, and stability under pasting by Artzner et al. (2002). Delbaen (2002) characterizes consistency (or multiplicative stability) in terms of martingale theory.

To my knowledge, this is the first paper to derive Eq. (1) from a set of axioms.[3] A different axiomatic approach can be found in Wang (2002). The main difference is that Wang does not assume translation invariance; thus, the corresponding class of risk measures need not be coherent (nor convex). A very interesting axiomatic approach to distribution-invariant dynamic risk measures can be found in the recent working paper by Weber (2004). Cvitanic and Karatzas (1999) assume that a complete financial market is given and define the risk of a position as the highest expected shortfall under some set of probability measures when the position is completely hedged given a fixed initial capital.

The next section defines dynamic risk measures and exposes my set of axioms. Section 2 defines families of conditional probability measures and the consistency property for these sets. Section 3 contains the main representation result. Stochastic interest rates are being discussed in Section 4. Appendices B and C gather some proofs.

1. Dynamic risk measures, coherency, and consistency

Consider a sequence of time periods $t = 0, \dots, T$, a finite set of states of the world Ω and a sequence of random variables $X_t : \Omega \to \mathbb{R}$, $t = 0, \dots, T$. X_t is to be understood as the information revealed at time t. Accordingly, the corresponding information filtration is $\mathscr{F}_t = \sigma(X_1, \dots, X_t)$, $t = 1, \dots, T$ and $\mathscr{F}_0 = \{\emptyset, \Omega\}$.[4] A position $D = (D_t)$ is a $(\mathscr{F}_t)$-adapted process, to be interpreted as a sequence of random payments D_t at time t. The set of all positions is denoted $\mathscr{D}$. For Sections 1 and 3, $r > -1$ is a fixed, exogenous interest rate. An extension to stochastic interest rates is given in Section 4.

Given this general dynamic model, I propose the following definition of dynamic risk measures.

Definition 1. A dynamic risk measure $\rho = (\rho_t)_{t=0,\dots,T}$ consists of mappings $\rho_t : \mathscr{D} \times \Omega \to \mathbb{R}$ such that

(1) for all $D, D' \in \mathscr{D}$ and t: if $D(s, \omega) = D'(s, \omega)$ for all $s \geqslant t$ and all $\omega \in \Omega$, then $\rho_t(D, \omega) = \rho_t(D', \omega)$; (independence of the past),
(2) for all t and $D \in \mathscr{D}$, $\rho_t(D, \cdot)$ is $\mathscr{F}_t$-measurable (adapted),
(3) ρ_t is monotone: $D \geqslant D'$ implies $\rho_t(D, \omega) \leqslant \rho_t(D', \omega)$,

[3] Artzner et al. (2002) adapt the static coherent risk measure axioms to an extended state space including states of nature and points in time to obtain a representation of the form

$$\inf_{A \in \mathscr{A}} \mathbb{E}^{\mathbb{P}} \left[\sum_{t=0}^{T} X_t (A_t - A_{t-1}) \right]$$

for one probability measure $\mathbb{P}$ and a set of positive increasing adapted processes $\mathscr{A}$. Moreover, they study the consistency property in simple examples and compare it to dynamic programming issues (compare Lemma 2). This approach has subsequently been generalized to continuous-time models in Cheridito et al. (2003). See also Roorda et al. (2003) and Scandolo (2003).

[4] Note that every finite dynamic model is included in this setup. In particular, every finite tree can be modeled that is done here.

(4) ρ_t is translation invariant with respect to predictable income streams: let Z be $\mathscr{F}_t$-measurable, set $D(s,w) \triangleq Z(\omega)1_{\{\tau\}}(s)$ for some $\tau \geqslant t$. Then for all positions D' and all $\omega \in \Omega$

$$\rho_t(D' + D, \omega) = \rho_t(D', \omega) - \frac{Z(\omega)}{(1+r)^{\tau - t}}.$$

(Predictable translation invariance.)

Independence of the past is a structural property of dynamic risk measures. Past payments are sunk and should not influence the assessment how risky the remaining future payments are. Also, the risk measure cannot depend on information to be revealed in the future; in other words, the dynamic risk measure has to be adapted. Monotonicity is certainly a reasonable requirement, whereas predictable translation invariance might need some explanation. Here is why I think that this property is reasonable. If one views the risk of a position as the amount of money one has to add to the position to make it acceptable, then it is clear that after adding, say, 1000$ to a position, the amount of money needed to make the position acceptable, is reduced by 1000$. This is translation invariance. In the present dynamic setting, this translation invariance carries over to payments that are known in advance (predictable). Adding 1000$ at time $t + \tau$ to a position D is equivalent to adding the present value of 1000 τ–$ at time t. More generally, adding a predictable position to a given position corresponds to adding the present value of the predictable position in t; thus, the risk of a given position should be reduced by the present value of the predictable position.

In this and the following section, I take a constant interest rate $r > -1$ as given. That is, the agent has a certain reference return r which he uses for calculating present values of deterministic income streams. I do think that such an interest rate exists in most relevant circumstances; in today's well developed markets, every individual, bank or regulatory agency has at least access to some kind of credit market with a predictable interest rate. Here, I assume for simplicity that the interest rate is not only predictable, but even constant. The main theorem is still valid, however, with stochastic interest rates as long as for every point in time discount factors (or zero coupon bonds) for all relevant maturities are available. This is being shown in Section 4.

Axiom 1. A dynamic risk measure (ρ_t) is called coherent if every ρ_t is homogeneous and subadditive. Formally, for all $D, D' \in \mathscr{D}$, $\lambda > 0$, $t = 0, \ldots, T$, and $\omega \in \Omega$ $\rho_t(\lambda D, \omega) = \lambda \rho_t(D, \omega)$ and $\rho_t(D + D', \omega) \leqslant \rho_t(D, \omega) + \rho_t(D', \omega)$ hold true.

Coherency says that mergers do not increase risk, yet mergers of identical positions do not reduce risk either. I follow ADEH in imposing coherency since this is definitely the aim of the present paper. Coherency implies convexity, and the latter is certainly economically meaningful, as it corresponds to the fact that diversification reduces risk. [5]

[5] It might be tempting to assume only convexity as in Föllmer and Schied (2002a). An extension to the dynamic framework under this weaker axiom is not attempted here. An axiomatic analysis of convex, conditioned risk measures has been carried out subsequently to this paper in Detlefsen (2003) and Scandolo (2003).

Axiom 2. A dynamic risk measure ρ is dynamically consistent iff for all times $t = 0,\dots,T-1$ and positions $D, D' \in \mathscr{D}$ with $D_t = D'_t$ the following holds true: $\rho_{t+1}(D,\omega) = \rho_{t+1}(D',\omega)$ for all $\omega \in \Omega$ implies $\rho_t(D,\omega) = \rho_t(D',\omega)$ for all $\omega \in \Omega$.

Dynamic consistency as just defined states in particular that if a position leads to an uncertain payment in, say, December of 2004 (and only then), and I know that I will accept that position in November of 2004 whatever state of the world prevails, then I should accept that position also in 2003. I hope that the reader agrees with me that such consistency is desirable.

Finally, I introduce

Axiom 3. For $t \in \{0,\dots,T\}$ and $\omega \in \Omega$ denote by $1_{\{(t,\omega)\}}$ the indicator function that has the value 1 in (t,ω) and the value 0 otherwise. A dynamic risk measure (ρ_t) is relevant if every loss which is not excluded by the given history carries positive risk: for all t, all histories $\xi_t \in \mathscr{H}_t$, all $\omega, \bar{\omega} \in \Omega(\xi_t)$, and all $\tau \geqslant t$, one has

$$\rho_t(-1_{\{(\tau,\bar{\omega})\}})(\omega) > 0.$$

The preceding axiom extends the relevance axiom in ADEH to the dynamic framework. It states that every path which is not excluded by the known history is relevant in the sense that a possible loss on that path carries positive risk.

2. Consistent sets of probability measures

As sets of (conditional) probability measures will play an important role, I introduce some relevant notation as well as the crucial consistency property for such sets before formulating the representation theorem.

A history up to time t is given by a sequence $\xi_t = (x_1,\dots,x_t)$ of realizations of X_s, $s = 1,\dots,t$. The set of all histories up to time t is denoted $\mathscr{H}_t = \mathrm{range}((X_1,\dots,X_t))$. The empty history at time 0 is denoted $\emptyset$. The set of all possible continuations after some history ξ_t is given by

$$\Omega(\xi_t) \triangleq \{\omega \in \Omega : (X_1,\dots,X_t)(\omega) = \xi_t\}.$$

$\Delta(\xi_t)$ denotes the set of all probability measures on $\Omega(\xi_t)$. A probability measure on $\Omega(\xi_t)$ is to be interpreted as the conditional distribution of the random vector $(X_{t+1},\dots,X_T)$ given that $(X_1,\dots,X_t) = \xi_t$.

Definition 2. Assume that for all $t = 0,\dots,T-1$, every history $\xi_t \in \mathscr{H}_t$, $\mathscr{Q}^{\xi_t} \subset \Delta(\xi_t)$ is a closed and convex set of (conditional) probability measures. The collection $(\mathscr{Q}^{\xi_t})$ is called a family of conditional probability measures.

(i) Fix $\xi_t \in \mathscr{H}_t$. Choose for all x_{t+1} with $(\xi_t, x_{t+1}) \in \mathscr{H}_{t+1}$ a measure $Q^{(\xi_t,x_{t+1})} \in \mathscr{Q}^{(\xi_t,x_{t+1})}$. Moreover, let $R^{\xi_t} \in \mathscr{Q}^{\xi_t}$. The *composite probability measure* $Q^{(\xi_t,X_{t+1})}R^{\xi_t} \in \Delta(\xi_t)$ is defined for sets $A \subset \Omega(\xi_t)$ via

$$Q^{(\xi_t,X_{t+1})}R^{\xi_t}(A) \triangleq \sum_{x_{t+1}} Q^{(\xi_t,x_{t+1})}(A \cap \{X_{t+1}=x_{t+1}\})R^{\xi_t}(X_{t+1}=x_{t+1}), \tag{2}$$

where the sum runs over all x_{t+1} with $(\xi_t,x_{t+1}) \in \mathscr{H}_{t+1}$. $\mathscr{Q}^{(\xi_t,X_{t+1})}\mathscr{Q}^{\xi_t}$ denotes the set of all these probability measures.

(ii) The family of conditional probability measures $(\mathscr{Q}^{\xi_t})$ is called consistent iff

$$\mathscr{Q}^{\xi_t} = \mathscr{Q}^{(\xi_t,X_{t+1})}\mathscr{Q}^{\xi_t}$$

for all $\xi_t \in \mathscr{H}_t$ and $t=0,\dots,T-1$.

(iii) A set of probability measures $\mathscr{P} \subset \Delta$ is called consistent iff, for all t, the induced family of conditional probability measures $(\mathscr{P}^{\xi_t})$ with

$$\mathscr{P}^{\xi_t} = \{P[\,\cdot\,|(X_1,\dots,X_t)=\xi_t] : P \in \mathscr{P}, P[(X_1,\dots,X_t)=\xi_t] > 0\}$$

is consistent.

Example 1. Two trivial yet important examples of consistent sets of probability measures are the set Δ of all probability measures on Ω and the singleton $\mathscr{P}=\{P\}$ for one probability measure P. They correspond to the worst case risk measure and the mean loss risk measure, resp. See also Example 4.

Example 2. Suppose that a finite number $K+1$ of financial assets with price processes S^k, $k=0,1,\dots,K$ and a reference probability P are fixed. Let asset S^0 be a riskless bond with interest rate r, that is, $S^0(t)=\exp(rt)$, $t=0,\dots,T$. Denote by $\bar{S}^k(t) = S^k(t)/S^0(t)$, $(t=0,\dots,T,\ k=1,\dots,K)$ the K discounted risky assets. If the financial market is arbitrage free, the Fundamental Theorem of Asset Pricing tells us that the set of equivalent martingale measures

$$\mathscr{P} = \{Q \in \Delta : Q \sim P, \bar{S}^k \text{ is a } Q\text{-martingale for all } k=1,\dots,K\}$$

is not empty. As one can easily check, the closure of $\mathscr{P}$ is consistent.

Remark 1. Note that the converse to Example 2 is not true. Not every consistent set of probability measures is the set of equivalent martingale measures for some financial market. Just take the one period model with two states of the world, one riskless asset $S^1 \equiv 1$, and another asset S^2. Assume furthermore that no arbitrage opportunities exist. Then either all probabilities in the interior of Δ are equivalent martingale measures (if S^2 is riskless), or only one. On the other hand, every set of probability measures is trivially consistent in a one period world.

Example 3 (The consistent hull). As arbitrary intersections of consistent sets of probability measures are consistent, we can speak of the *consistent hull* $\mathscr{P}^{\text{cons}}$ of a set of probability measures $\mathscr{P}$, being the smallest consistent set containing $\mathscr{P}$. The consistent hull can be obtained by a backward induction procedure. The procedure can be easily illustrated with two periods. Define for every possible realization x_1 of the first period, the set $\mathscr{P}^{x_1} = \{P[\,\cdot\,|X_1 = x_1] : P \in \mathscr{P},\ P(X_1 = x_1) > 0\}$ of conditional distributions for the second period random variable X_2 given that $X_1 = x_1$. Then, for every choice of conditionals $Q^{x_1} \in \mathscr{P}^{x_1}$ and marginal distribution $R \in \mathscr{P}$, one can derive the joint distribution of X_1 and X_2 by the formula of total probability (2). One obtains the consistent hull $\mathscr{P}^{\mathrm{cons}}$ by pasting together conditionals of X_2 from $(\mathscr{P}^{x_1})$ and marginal distributions of X_1 from $\mathscr{P}$ in every possible way.

Before returning to risk measures, let us collect an important property of consistent sets of probability measures.

Lemma 1 (Backward induction and consistency). (1) *A family of conditional probability measures $(\mathscr{Q}^{\xi_t})$ is consistent if and only if for all $\xi_t \in \mathscr{H}_t$ and all random variables $Z : \Omega \to \mathbb{R}$ the recursive relation*

$$\min_{Q \in \mathscr{Q}^{\xi_t}} \int_{\Omega(\xi_t)} Z \,\mathrm{d}Q = \min_{Q \in \mathscr{Q}^{\xi_t}} \int_{\Omega(\xi_t)} \min_{R \in \mathscr{Q}^{(\xi_t, X_{t+1}(\omega))}} \int_{\Omega(\xi_t, X_{t+1}(\omega))} Z(\omega') R(\mathrm{d}\omega') Q(\mathrm{d}\omega) \quad (3)$$

holds true.

(2) *A set of probability measures $\mathscr{P} \subset \Delta$ is consistent iff, for all t and for all random variables $Z : \Omega \to \mathbb{R}$, one has*

$$\min_{P \in \mathscr{P}} \mathbb{E}^P[Z|\mathscr{F}_t] = \min_{P \in \mathscr{P}} \mathbb{E}^P \left[\min_{Q \in \mathscr{P}} \mathbb{E}^Q[Z|\mathscr{F}_{t+1}] \middle| \mathscr{F}_t \right].$$

Proof. See Appendix B. □

The preceding lemma shows that consistency of a family corresponds exactly to the usual logic of backward induction. If one first minimizes the conditional expectation of a random variable Z given a certain history x_1 and minimizes then the expected value of this function over all possible values x_1, one obtains the same value as if one minimizes the expected value of the random variable ex ante.

The following lemma shows that a consistent family of conditional probability measures with full support is generated by its initial set.

Lemma 2. *Assume that $(\mathscr{Q}^{\xi_t})$ is a consistent family of conditional probability measures with full support. Then the family is determined by its initial set $\mathscr{Q}^{\emptyset}$:*

$$\mathscr{Q}^{\xi_t} = \{Q[\,\cdot\,|(X_1, \ldots, X_t) = \xi_t] : Q \in \mathscr{Q}^{\emptyset},\ Q[(X_1, \ldots, X_t) = \xi_t] > 0\}$$

for all histories $\xi_t \in \mathscr{H}_t$, $t = 0, \ldots, T-1$.

Proof. See Appendix B. □

The full support assumption is needed here to ensure that the sets of conditional probability measures are not empty for all histories. Consistency implies that the agent obtains the family of conditional probability measures by updating every measure of the initial set of probability measures that puts positive probability on the given history.

3. The representation theorem

After these preliminaries, the main representation theorem can be stated and proved. Recall that a set of probability measures $\mathscr{P}$ on a measurable space $(A, \mathscr{A})$ has *full support* if the union of all supports in $\mathscr{P}$ is the whole set of states A.

Theorem 1. *The following assertions are equivalent for a dynamic risk measure* ρ:

(1) ρ *is coherent, dynamically consistent, and relevant,*
(2) *there exists a closed, convex, and consistent set of probability measures* $\mathscr{P} \subset \Delta$ *with full support such that*

$$\rho_t(D) = \max_{P \in \mathscr{P}} \mathbb{E}^P \left[- \sum_{s=t}^{T} \frac{D_s}{(1+r)^{s-t}} \middle| \mathscr{F}_t \right]. \tag{4}$$

Proof. See Appendix C. □

The above theorem shows that every dynamic coherent risk measure corresponds to a choice of a convex, closed, and consistent set $\mathscr{P}$ of probability measures with full support. I list a number of examples that extend the examples of consistent sets of probability measures in Section 2 to dynamic risk measures.

Example 4. Let $P \in \Delta$ have full support. Then the conditional mean loss

$$\rho_t(D) = \mathbb{E}^P \left[- \sum_{s=t}^{T} \frac{D_s}{(1+r)^{s-t}} \middle| \mathscr{F}_t \right]$$

forms a consistent and relevant dynamic coherent risk measure. The choice $\mathscr{P} = \Delta$ leads to the *worst case risk measure*

$$\rho_t(D) = \max_{\omega \in \Omega(X_1, \ldots, X_t)} - \sum_{s=t}^{T} \frac{D_s}{(1+r)^{s-t}}.$$

Since Δ is consistent, so is the worst case risk measure.

Example 5 (Superhedging risk measure). Suppose that $\mathscr{P}$ is the closure of the set of equivalent martingale measures for some financial market as in Example 2. The *superhedging risk measure*

$$\rho_t(D) = \max_{P \in \mathscr{P}} \mathbb{E}^P \left[- \sum_{s=t}^{T} \frac{D_s}{(1+r)^{s-t}} \middle| \mathscr{F}_t \right]$$

is consistent (and coherent).

Example 6 (Consistent envelope of a risk measure). Let ρ be a coherent risk measure. By the ADEH theorem, ρ corresponds to some set of probability measures $\mathscr{P}$. Usually, the straightforward extension of ρ to a dynamic setting via (4) will not lead to a consistent dynamic risk measure. However, by passing to the consistent hull $\mathscr{P}^{\text{cons}}$ of

$\mathscr{P}$ (see Example 3), one gets a consistent dynamic risk measure, the *consistent envelope* ρ^{cons} of ρ that dominates ρ.

I conclude this section with a remark on dynamic consistency and backward induction or dynamic programming. As the reader recognizes easily from the proof, or directly from Lemma 1, a dynamic coherent risk measure satisfies the recursive relation

$$\rho_t(D) = \max_{\mathbb{P}\in\mathscr{P}} \mathbb{E}^{\mathbb{P}}[-D_t + (1+r)^{-1}\rho_{t+1}(D)|\mathscr{F}_t].$$

This equation greatly simplifies the analysis of dynamic optimization problems related to dynamic risk measures and is also of importance for numerical implementations since it allows to calculate dynamic risks via backward induction. Moreover, one derives easily from this equation that the process $(1+r)^{-t}\rho_t(D) - \sum_{s=0}^{t-1}(1+r)^{-s}D_s$ is a supermartingale for all $P\in\mathscr{P}$. I record this reasoning in the following corollary.

Corollary 1. *Let (ρ_t) be a dynamically consistent, coherent, and relevant dynamic risk measure with associated set of probability measures $\mathscr{P}$. For $D\in\mathscr{D}$ the process*

$$S_t \triangleq \frac{\rho_t(D)}{(1+r)^t} - \sum_{s=0}^{t-1}\frac{D_s}{(1+r)^s}$$

is a supermartingale under every $P\in\mathscr{P}$. Moreover, for some $P^*\in\mathscr{P}$, S is a P^*-martingale.

Proof. Fix $D\in\mathscr{D}$ and set $Z \triangleq \sum_{s=0}^{T} D_s/(1+r)^s$. Note that

$$S_t = -\min_{P\in\mathscr{P}} \mathbb{E}^P[Z|\mathscr{F}_t].$$

Let $P^0\in\mathscr{P}$. From Lemma 1,

$$\begin{aligned} S_t &= -\min_{P\in\mathscr{P}} \mathbb{E}^P\left[\min_{Q\in\mathscr{P}} \mathbb{E}^Q[Z|\mathscr{F}_{t+1}]\middle|\mathscr{F}_t\right] \\ &= \max_{P\in\mathscr{P}} \mathbb{E}^P[S_{t+1}|\mathscr{F}_t] \geqslant \mathbb{E}^{P^0}[S_{t+1}|\mathscr{F}_t]. \end{aligned} \tag{5}$$

Hence, S is a P^0-supermartingale.

From (5), one sees that for every $t=0,\ldots,T$ and history $\xi_t\in\mathscr{H}_t$, there exists a probability $P^{\xi_t}\in\mathscr{P}$ such that $S_t = E^{P^{\xi_t}}[S_{t+1}|(X_1,\ldots,X_t)=\xi_t]$. Define the probability measure P^* by setting for $\omega\in\Omega$ with $X_t(\omega)=x_t, t=1,\ldots,T$,

$$P^*(\{\omega\}) = P^{\emptyset}(X_1=x_1)P^{x_1}(X_2=x_2)\cdots P^{(x_1,\ldots,x_{T-1})}(X_T=x_T).$$

It is straightforward to check that with this definition,

$$E^{P^*}[S_{t+1}|X_1,\ldots,X_t)=\xi_t] = E^{P^{\xi_t}}[S_{t+1}|X_1,\ldots,X_t)=\xi_t].$$

Hence, S is a P^*-martingale. Since $\mathscr{P}$ is consistent, $P^*\in\mathscr{P}$. This concludes the proof. □

4. Stochastic interest rates

In this section, let us assume that interest rates are stochastic. For every maturity $\tau = 1, \ldots, T$, there exists a zero coupon bond B^τ that pays off 1$ at time τ. The (possibly random) price of the bond with maturity τ at time t is denoted by B_t^τ. Since the short rate is no longer known in advance, one has to change the definition of predictable translation invariance (Definition 1(4)). A predictable payment Z at time τ has the present value ZB_t^τ at time t. Therefore, the natural definition for predictable translation invariance is

Definition 1(4′). Let Z be $\mathscr{F}_t$-measurable; set $D(s,w) \triangleq Z(\omega)1_{\{\tau\}}(s)$ for some $\tau \geqslant t$. Then for all positions D' and all $\omega \in \Omega$

$$\rho_t(D' + D, \omega) = \rho_t(D', \omega) - Z(\omega)B_t^\tau(\omega).$$

In this case, the following theorem holds true.

Theorem 2. *The following assertions are equivalent for a dynamic risk measure* ρ:

(1) ρ *is coherent, dynamically consistent, and relevant,*
(2) *there exists a closed, convex, and consistent set of probability measures* $\mathscr{P} \subset \Delta$ *with full support such that*

$$\rho_t(D) = \max_{P \in \mathscr{P}} \mathbb{E}^P \left[- \sum_{s=t}^{T} D_s B_t^s \middle| \mathscr{F}_t \right]$$

The proof is a straightforward adaptation of the proof of Theorem 1.

5. Conclusion

Consistency is a necessary requirement for rational dynamic decision making. The present paper has characterized consistent dynamic coherent risk measures. Beyond this contribution for the field of financial risk measurement, the paper can be applied to a variety of other contexts in which robust extensions of the usual expectation operator have recently been studied. Chateauneuf et al. (1996) introduce risk measures as price functionals in markets with transaction costs. Wang et al. (1997) use an axiomatic approach to insurance pricing that is in many respects very similar to Artzner et al. (1999). The present paper provides also a framework for dynamic extensions of these papers. In macroeconomics, robust decision making has been studied by Anderson et al. (2000). Last but not the least, the dynamic version of minimax expected utility in Epstein and Schneider (2003) is a form of ‘expected’ utility where the expectation is replaced with a consistent dynamic coherent risk measure.

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Appendix A. The ADEH theorem

In the proof of the representation theorem, I use the following version of the ADEH theorem.

Theorem A.1. *Let $(A, \mathscr{A})$ be a finite measurable space and ρ a coherent risk measure on $(A, \mathscr{A})$ with interest rate $R > 0$. Then ρ can be represented as*

$$\rho(D) = \max_{P \in \mathscr{P}} \mathbb{E}^{\mathbb{P}} \left(-\frac{D}{R} \right)$$

for a closed and convex set of probability measures $\mathscr{P}$ on $(A, \mathscr{A})$.

This may seem stronger than the ADEH theorem as I replace the supremum by the maximum and the (arbitrary) set of probability measures by its convex closure here. But as the reader easily checks, there is no loss of generality in doing so. A proof is given by Föllmer and Schied (2002b, Corollary 14).

Appendix B. Proof of the lemmas

In the proof of the lemmas, I will use the following fact which is an immediate consequence of the separation theorem for convex sets:

Lemma B.1. *Two closed and convex sets $\mathscr{Q}, \mathscr{Q}' \subset \Delta$ are equal if and only if for all random variables $Z : \Omega \to \mathbb{R}$ one has*

$$\min_{Q \in \mathscr{Q}} \int_{\Omega} Z \, \mathrm{d}Q = \min_{Q \in \mathscr{Q}'} \int_{\Omega} Z \, \mathrm{d}Q.$$

Moreover, I shall need the following lemma.

Lemma B.2. *Suppose that for every history ξ_t, the set $\mathscr{Q}^{\xi_t} \subset \Delta(\xi_t)$ is closed and convex. Then the sets $\mathscr{Q}^{(\xi_{t-1}, X_t)} \mathscr{Q}^{\xi_{t-1}}$ are also closed and convex.*

Proof. From the formula of total probability (2), closedness follows immediately. For convexity, take, for simplicity, $t = 1$, and choose $Q_0, Q_1 \in \mathscr{Q}^{X_1} \mathscr{Q}$ and $0 < \lambda < 1$. One has to show that $Q_2 \triangleq \lambda Q_0 + (1 - \lambda) Q_1 \in \mathscr{Q}^{X_1} \mathscr{Q}$. For that matter, choose $R_0^{x_1}, R_1^{x_1} \in \mathscr{Q}^{x_1}$ for all $x_1 \in \mathrm{range}(X_1)$ and $Q'_0, Q'_1 \in \mathscr{Q}$ such that

$$Q_i = R_i^{X_1} Q'_i, \quad i = 0, 1. \tag{B.1}$$

Set $Q_2' \triangleq \lambda Q_0' + (1-\lambda)Q_1' \in \mathcal{Q}$. For x_1 with $Q_2(X_1 = x_1) > 0$, set

$$\mu(x_1) \triangleq \frac{\lambda Q_0'(X_1 = x_1)}{Q_2'(X_1 = x_1)}.$$

Since $0 < \mu(x_1) < 1$ and $\mathcal{Q}^{x_1}$ is convex,

$$R_2^{x_1} \triangleq \mu(x_1)R_0^{x_1} + (1-\mu(x_1))R_1^{x_1} \in \mathcal{Q}^{x_1}.$$

The following calculation shows that $Q_2 = R_2^{X_1}Q_2'$, and hence $Q_2 \in \mathcal{Q}^{X_1}\mathcal{Q}$. For $A \subset \Omega$, set $A_{x_1} \triangleq A \cap \{X_1 = x_1\}$; then

$$\begin{aligned}
R_2^{X_1}Q_2'(A) &= \sum_{x_1} R_2^{x_1}(A_{x_1})Q_2'(X_1 = x_1) \\
&= \sum_{x_1} [\mu(x_1)R_0^{x_1}(A_{x_1}) + (1-\mu(x_1))R_1^{x_1}(A_{x_1})]Q_2'(X_1 = x_1) \\
&= \sum_{x_1} \lambda R_0^{x_1}(A_{x_1})Q_0'(X_1 = x_1) + \sum_{x_1} (1-\lambda)R_1^{x_1}(A_{x_1})Q_1'(X_1 = x_1) \\
&= \lambda Q_0(A) + (1-\lambda)Q_1(A) = Q_2(A),
\end{aligned}$$

where the last line uses (B.1). □

We can now tackle Lemma 1.

Proof of Lemma 1. For simplicity, let us consider the case $t = 0$. Let

$$K \triangleq \min_{Q\in\mathcal{Q}} \int_\Omega \min_{R\in\mathcal{Q}^{X_1(\omega)}} \int_{\Omega(X_1(\omega))} Z(\omega')R(\mathrm{d}\omega')Q(\mathrm{d}\omega),$$

and

$$L \triangleq \min_{Q\in\mathcal{Q}^{X_1}\mathcal{Q}} \int_\Omega Z\,\mathrm{d}Q.$$

I show below that $K = L$. Given this, (3) reduces to

$$\min_{Q\in\mathcal{Q}} \int_\Omega Z\,\mathrm{d}Q = \min_{Q\in\mathcal{Q}^{X_1}\mathcal{Q}} \int_\Omega Z\,\mathrm{d}Q,$$

and Lemmas B.1 and B.2 conclude the proof.

It remains to show that $K = L$. For every $x_1 \in \mathcal{H}_1$, choose

$$Q^{x_1} \in \arg\min_{R\in\mathcal{Q}^{x_1}} \int_{\Omega(x_1)} Z(\omega')R(\mathrm{d}\omega'),$$

and choose $Q \in \arg\min_{Q\in\mathcal{Q}} \int_\Omega \min_{R\in\mathcal{Q}^{X_1(\omega)}} \int_{\Omega(X_1(\omega))} Z(\omega')R(\mathrm{d}\omega')Q(\mathrm{d}\omega)$. Then

$$K = \int_\Omega \int_{\Omega(X_1(\omega))} Z(\omega')Q^{X_1(\omega)}(\mathrm{d}\omega')Q(\mathrm{d}\omega) = \int_\Omega Z\,\mathrm{d}(Q^{X_1}Q).$$

Since $Q^{X_1}Q \in \mathcal{Q}^{X_1}\mathcal{Q}$, $K \geqslant L$ follows. On the other hand, if $P = Q^{X_1}Q$ is such that $L = \int Z\,\mathrm{d}P$, then

$$L = \int_\Omega \int_{\Omega(X_1(\omega))} Z(\omega')Q^{X_1(\omega)}(\mathrm{d}\omega')Q(\mathrm{d}\omega)$$

$$\geqslant \int_\Omega \min_{R\in\mathcal{Q}^{X_1(\omega)}} \int_{\Omega(X_1(\omega))} Z(\omega')R(\mathrm{d}\omega')Q(\mathrm{d}\omega)$$

$$\geqslant K.$$

Therefore, $K = L$, and the proof is complete. □

Proof of Lemma 2. I show the claim for the initial step from $\mathcal{Q}$ to $\mathcal{Q}^{x_1}$. Induction does the rest. By consistency, every $Q \in \mathcal{Q}$ can be written as

$$Q = R^{X_1}R$$

for suitable $R^{x_1} \in \mathcal{Q}^{x_1}$ and $R \in \mathcal{Q}$. Assume $Q(X_1 = x_1) > 0$. From (2), it follows that

$$Q(\cdot|X_1 = x_1) = R^{x_1}(\cdot) \in \mathcal{Q}^{x_1},$$

and thus

$$\{Q(\cdot|X_1 = x_1) : Q \in \mathcal{Q},\ Q(X_1 = x_1, \ldots, X_t = x_t) > 0\} \subset \mathcal{Q}^{x_1}.$$

On the other hand, let $Q^{x_1} \in \mathcal{Q}^{x_1}$ be given. For every other possible realization $y_1 \neq x_1$ choose $Q^{y_1} \in \mathcal{Q}^{y_1}$. By the full support assumption, there is $R \in \mathcal{Q}$ with $R(X_1 = x_1) > 0$. Set $\bar{Q}(A) = Q^{x_1}(A \cap \{X_1 = x_1\})R(X_1 = x_1) + \sum_{y_1 \neq x_1} Q^{y_1}(A \cap \{X_1 = y_1\})R(X_1 = y_1)$. By consistency, $\bar{Q} \in \mathcal{Q}$ and, by construction, $Q^{x_1} = \bar{Q}(\cdot|X_1 = x_1)$. This shows that

$$Q^{x_1} \in \{Q(\cdot|X_1 = x_1, \ldots, X_t = x_t) : Q \in \mathcal{Q},\ Q(X_1 = x_1, \ldots, X_t = x_t) > 0\}. \qquad \square$$

Appendix C. Proof of the representation theorem

Proof of Theorem 1. Let us start with the easier implication from (2) to (1). All properties are either obvious or well known from Artzner et al. (1999) except dynamic consistency. So let $D, D' \in \mathcal{D}$ be given and fix some t. By predictable translation invariance, one can assume without loss of generality that $D_t = D'_t = 0$. Suppose further that $\rho_{t+1}(D, \omega) = \rho_{t+1}(D', \omega)$ for all $\omega \in \Omega$, that is

$$\min_{Q\in\mathcal{P}} \mathbb{E}^Q \left[\sum_{s=t+1}^{T} \frac{D_s}{(1+r)^{s-t-1}} \middle| \mathcal{F}_{t+1} \right] = \min_{Q\in\mathcal{P}} \mathbb{E}^Q \left[\sum_{s=t+1}^{T} \frac{D'_s}{(1+r)^{s-t-1}} \middle| \mathcal{F}_{t+1} \right]$$

for all ω. From this and $D_t = D'_t = 0$, one gets

$$\min_{P\in\mathcal{P}} \mathbb{E}^P \left[\min_{Q\in\mathcal{P}} \mathbb{E}^Q \left[\sum_{s=t}^{T} \frac{D_s}{(1+r)^{s-t}} \middle| \mathcal{F}_{t+1} \right] \middle| \mathcal{F}_t \right]$$

$$= \min_{P\in\mathscr{P}} \mathbb{E}^P \left[\min_{Q\in\mathscr{P}} \mathbb{E}^Q \left[\sum_{s=t}^{T} \frac{D'_s}{(1+r)^{s-t}} \middle| \mathscr{F}_{t+1} \right] \middle| \mathscr{F}_t \right].$$

Since $\mathscr{P}$ is consistent, Lemma 1 yields $\rho_t(D,\omega) = \rho_t(D',\omega)$.

Let us now consider the more difficult implication (1) to (2). Fix a time period t and a history $\xi_t \in \mathscr{H}_t$. Define the expanded state space $S(\xi_t) \triangleq \{t,\dots,T\} \times \Omega(\xi_t)$ and endow this space with the σ-field $\mathscr{O}(\xi_t)$ generated by all $D \in \mathscr{D}$ restricted to $S(\xi_t)$. Denote by $\mathscr{D}(\xi_t)$ the set of these restricted positions. By Definition 1(1) and 1(2), the mapping

$$\sigma : \mathscr{D}(\xi_t) \to \mathbb{R},$$

$$D \mapsto \rho_t(D)(\omega),$$

is well-defined and independent of ω. σ inherits monotonicity, subadditivity, and homogeneity from ρ_t. σ is translation invariant in the following sense. The mapping $D^\alpha \equiv \alpha \in \mathbb{R}$ corresponds to a sure payment of α in every period. Therefore, by iterated application of predictable translation invariance (Definition 1.4′)

$$\sigma(D' + D^\alpha) = \sigma(D') - \sum_{s=t}^{T} \frac{\alpha}{(1+r)^{s-t}}.$$

Let $R(t) \triangleq \left(\sum_{s=t}^{T} 1/(1+r)^{s-t}\right)^{-1}$. We have just shown that σ is a coherent risk measure on the space $(S(\xi_t), \mathscr{O}(\xi_t))$ with interest rate $R(t)$. The ADEH theorem (in the version stated in Appendix A) yields a convex and closed set $\mathscr{M}$ of probability measures on $(S(\xi_t), \mathscr{O}(\xi_t))$ with

$$\sigma(D) = \max_{m\in\mathscr{M}} \int_{S(\xi_t)} -\frac{D(s,\omega)}{R(t)}\, m(\mathrm{d}s, \mathrm{d}\omega). \tag{C.1}$$

The next step is to show that the marginal distribution $m(\{\tau\} \times \Omega(\xi_t))$ is the same for all $m \in \mathscr{M}$. To this end, fix $\tau \in \{t,\dots,T\}$ and consider the position that yields a certain payment of 1 at time τ:

$$D_s = \begin{cases} 1 & s = \tau, \\ 0 & \text{else} \end{cases}$$

for all $\omega \in \Omega(\xi_t)$. By predictable translation invariance, $\sigma(D) = -1/(1+r)^{\tau-t}$. On the other hand, from (C.1), it follows that $\sigma(D) = \max_{m\in\mathscr{M}} -R(t)^{-1} m(\{\tau\} \times \Omega(\xi_t))$. Similarly, with $D_s = -1_{\{\tau\}}(s)$, $s = t,\dots,T$, one concludes that

$$\max_{m\in\mathscr{M}} R(t)^{-1} m(\{\tau\} \times \Omega(\xi_t)) = \frac{1}{(1+r)^{\tau-t}},$$

and $m(\{\tau\} \times \Omega(\xi_t)) = R(t)/(1+r)^{\tau-t}$ for all $m \in \mathscr{M}$ follows. Therefore, we obtain $m = \sum_{s=t}^{T} m(\mathrm{d}\omega|s) R(t)/(1+r)^{s-t}\, \delta_s$ and

$$\sigma(D) = \max_{m\in\mathscr{M}} \int_{\Omega(\xi_t)} -\sum_{s=t}^{T} \frac{D(s,\omega)}{(1+r)^{s-t}}\, m(\mathrm{d}\omega|s).$$

The set of conditional distributions $\mathscr{P}|\tau \triangleq \{m(\cdot|\tau) : m \in \mathscr{M}\}$ does not depend on τ. This follows from predictable translation invariance and dynamic consistency, as I now show. Let $Z:\Omega\to\mathbb{R}$ be a random variable that is known at time $\tau>t$, that is, Z is $\mathscr{F}_\tau$-measurable. Set $D(s,\omega)=Z(\omega)(1+r)^\tau 1_{\{\tau\}}(s)-Z(\omega)(1+r)^T 1_{\{T\}}(s)$. Predictable translation invariance says that

$$\rho_\tau(D,\omega)=0$$

for all $\omega\in\Omega$. By dynamic consistency, one obtains $\rho_t(D,\omega)=0$ for all ω, and thus $\sigma(D)=0$, or

$$0=\sup_{m\in\mathscr{M}}\int_{\Omega(\xi_t)} Z(\omega)[m(\mathrm{d}\omega|\tau)-m(\mathrm{d}\omega|T)].$$

Since this holds true for all $\mathscr{F}_\tau$-measurable random variables Z, it follows (perhaps invoking a separation argument, see also Appendix B) that $m(\cdot|\tau)=m(\cdot|T)$ on $\mathscr{F}_\tau\cap\Omega(\xi_t)$. Therefore, the conditional distributions can all be obtained from the conditional distribution at time T. Setting $\mathscr{P}(\xi_t)\triangleq\{m(\cdot|T):m\in\mathscr{M}\}$, one gets therefore

$$\rho_t(D)=\max_{P\in\mathscr{P}(\xi_t)}\sum_{s=t}^{T}\int_{\Omega(\xi_t)}-\frac{D_s}{(1+r)^{s-t}}\,\mathrm{d}P.$$

Note that $\mathscr{P}(\xi_t)$ inherits closedness and convexity from $\mathscr{M}$. Moreover, from the Relevance Axiom, one gets immediately that $\mathscr{P}(\xi_t)$ has full support.

Finally, I show that $(\mathscr{P}(\xi_t))$ is consistent. Let Z be a random variable, and set $D(s,\omega)=Z(\omega)\,(1+r)^{T-t}\,1_{\{T\}}(s)$. Define $D'(s,\omega)=-\rho_{t+1}(D,\omega)1_{\{t+1\}}(s)$. By predictable translation invariance, $\rho_{t+1}(D',\omega)=\rho_{t+1}(D,\omega)$ for all ω. Dynamic consistency yields then

$$\rho_t(D',\omega)=\rho_t(D,\omega),$$

or

$$\begin{aligned}&\sup_{P\in\mathscr{P}(\xi_t)}\int_{\Omega(\xi_t)}-Z\,\mathrm{d}P\\&\quad=\sup_{P\in\mathscr{P}(\xi_t)}\int_{\Omega(\xi_t)}\frac{\rho_{t+1}(D)}{1+r}\,\mathrm{d}P\\&\quad=\sup_{P\in\mathscr{P}(\xi_t)}\int_{\Omega(\xi_t)}\sup_{Q\in\mathscr{P}(\xi_t,X_{t+1}(\omega))}\int_{\Omega(\xi_t,X_{t+1}(\omega))}-Z(\omega')Q(\mathrm{d}\omega')P(\mathrm{d}\omega).\end{aligned}$$

By Lemma 1, consistency follows. This concludes the proof. □

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